Exercise 33

Find an equation for the plane containing the point $(1, 0, 1)$ and the line
$\mathbf{l}(t) = (1, 2, -1) + t(1, 0, 5)$.

Solution

The equation for a plane is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0,$$

where $\mathbf{n}$ is a vector normal to the plane and $\mathbf{r}_0$ is the position vector for any point in the plane. In order to get $\mathbf{n}$, take the cross product of the direction vector $(1, 0, 5)$ and the displacement vector,

$$(1, 0, 1) - (1, 2, -1) = (0, -2, 2).$$

Doing so gives

$$\mathbf{n} = (1, 0, 5) \times (0, -2, 2) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 0 & 5 \\ 0 & -2 & 2 \end{vmatrix} = (0+10)\hat{\mathbf{x}} - (2-0)\hat{\mathbf{y}} + (-2-0)\hat{\mathbf{z}} = 10\hat{\mathbf{x}} - 2\hat{\mathbf{y}} - 2\hat{\mathbf{z}} = (10, -2, -2).$$

Either of the position vectors, $(1, 0, 1)$ or $(1, 2, -1)$, will do for $\mathbf{r}_0$. Choose $\mathbf{r}_0 = (1, 0, 1)$.

$$(10, -2, -2) \cdot (x - 1, y - 0, z - 1) = 0$$

$$10(x - 1) - 2(y - 0) - 2(z - 1) = 0$$

$$10x - 10 - 2y - 2z + 2 = 0$$

$$10x - 2y - 2z = 8$$

$$5x - y - z = 4$$